TWO-QUARK BOUND STATES: MESON SPECTRA AND REGGE
TRAJECTORIES

G. Ganbold\textsuperscript{1,2†} and G.V. Efimov\textsuperscript{1}

(1) Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna
(2) Institute of Physics and Technology, MAS, Ulaanbaatar
† E-mail: ganbold@thsun1.jinr.ru

Abstract

We consider simple relativistic quantum field models with the Yukawa interaction of quarks and gluons. In the presence of the analytic confinement there exist bound states of quarks and gluons at relatively weak coupling. By using a minimal set of free parameters (the quark masses and the confinement scale) and involving the mass-dependent coupling constant of QCD we satisfactorily explain the observed meson spectra and the asymptotically linear Regge trajectories.

1 Introduction

The conventional theoretical description of colorless hadrons within the QCD implies they are bound states of quarks and gluons considered under the color confinement. A realisation of the confinement is developed in [1, 2] by using an assumption that QCD vacuum is realized by the self-dual homogeneous vacuum gluon field. Accordingly, this can lead to the quark and gluon confinement as well as a necessary chiral symmetry breaking. Hereby, propagators of quarks and gluons in this field are entire analytic functions in the $p^2$-complex plane, i.e. the analytic confinement takes place. On the other hand, there exists a prejudice to the idea of the analytic confinement (for example, [3]).

Therefore, it seems reasonable first to consider simple quantum field models in order to investigate only qualitative effects. In our earlier paper [4] we considered simple scalar-field models to clarify the ”pure role” of the analytic confinement in formation of the hadron bound states. In doing so, we have demonstrated just a mathematical sketch of calculations of the mass spectrum for the ”mesons” consisting of two ”scalar quarks” by omitting some quantum degrees of freedom such as the spin, color and flavor. These models gave a quite reasonable sight to the underlying physical principles of the hadron formation mechanism and the spectra, and have resulted in qualitative descriptions of ”mesons”, their Regge trajectories and ”glueballs” [5].

In this paper we present a more realistic extension of our earlier consideration by taking into account the omitted quantum characteristics associated with color, flavor and spin degrees of freedom for constituent quarks and gluons. Below we calculate the mass spectrum of pseudoscalar and vector mesons as well as glueballs.
2 The Model

Conventionally, considering this problem within QCD one deals with complicated and elaborate calculations, because the confinement is achieved as a result of strong interaction – by involving high-order corrections [6]. Then, there arises a problem of correct and effective summation of these contributions.

On the other hand, the use of a QFT method is effective when the coupling is not large. Then, lower orders of a perturbative technique can result in satisfactory accurate estimates (e.g., in QCD) of observables. Particularly, one can effectively use the one-gluon exchange approximation.

We consider a relativistic physics and for the hadronization processes use the Bethe-Salpeter equation. Because, when the binding energy is not negligible the relativistic corrections are considerable. Of course, in doing so, we use a minimal set of free parameters.

Consider the quark and gluon fields: $\Psi_{a}(x) = \Psi_{\text{color, spin, flavour}}(x)$, $\phi_{a}(x) = \phi_{\text{color}}(x)$

The Lagrangian of the Yukawa-type interaction reads:

$$L(x) = \left(\bar{\Psi}_{a}^{i} [S^{-1}]^{ij}_{\alpha\beta} \Psi_{j}^{\beta}\right) + \frac{1}{2} \left(\phi_{a}^{\mu} [D^{-1}]^{ab}_{\mu\nu} \phi_{b}^{\nu}\right) + g \left(\bar{\Psi}_{a}^{i} \left(\Gamma^{a}ight)^{ij}_{\alpha\beta} \Psi_{j}^{\beta}\right) + h \Lambda \left(\phi_{a}^{\mu} \phi_{b}^{\nu} F_{\mu\nu}^{a}\right) f^{abc}$$

$$\left(\Gamma^{a}ight)^{ij}_{\alpha\beta} = \left(i\gamma^{\mu}\right)^{ij}_{\alpha\beta}, \quad F_{\mu\nu}^{a} = \partial_{\mu} \phi_{\nu}^{a} - \partial_{\nu} \phi_{\mu}^{a}$$

We guess that the matrix elements of hadron processes at large distance (in the confinement region) are in fact integrated characteristics of quark (gluon) propagators and their vertices. Their tiny detailed behaviors may be not so important, but important is to take into account the correct symmetry features.

Our aim is to find the most plain forms of these propagators which keep the essential properties and result in a qualitative and semi-quantitative description of the hadron spectra. For this purpose, we consider the following propagators:

$$D_{\mu\nu}^{ij}(x) = \delta_{ab} \delta_{\mu\nu} \frac{\Lambda^{2}}{(4\pi)^{2}} e^{-x^{2}\Lambda^{2}/4}$$

$$\tilde{S}_{\alpha\beta}(\hat{p}) = \delta_{ij} \frac{1}{m^{2}} \left[i\hat{p} + m(1 + \gamma_{5}\omega(m))\right]_{\alpha\beta} e^{-\frac{x^{2} + m^{2}}{2\Lambda^{2}}}$$

where $\Lambda$ - the confinement energy scale, $m$ - the quark mass and

$$\omega(m) = \int_{0}^{1} du \left(\frac{1 - u}{1 + u}\right)^{\frac{n-2}{2}} \frac{u^{2}}{1 - u^{2}} \left\{\int_{0}^{1} du \left(\frac{1 - u}{1 + u}\right)^{\frac{n-2}{2}} \frac{1}{1 - u^{2}}\right\}^{-1}$$

These are entire analytic functions and represent reasonable approximations of the exact quark and gluon propagators.

Consider the partition function and take an explicit integration over $\phi$-variable:

$$Z = \int \delta \bar{\Psi} \delta \Psi e^{-\mathcal{L}_{F}[\bar{\Psi}, \Psi]} \int \delta \phi e^{-\mathcal{L}_{B}[\phi] - \mathcal{L}_{\text{int}}[\bar{\Psi}, \Psi, \phi]}$$

$$= \int \delta \bar{\Psi} \delta \Psi e^{-\mathcal{L}_{F}[\bar{\Psi}, \Psi] - \mathcal{L}_{2}[\bar{\Psi}, \Psi] - \mathcal{L}_{3}[\bar{\Psi}, \Psi] - \mathcal{L}_{\text{G}} + ...}$$
Then, the lowest bound states of quarks and gluons are expressed as follows.
Mesons:
\[ \mathcal{L}_2 = \frac{g^2}{2} \int \delta \phi \ e^{-\mathcal{L}_B[\phi]} \left( \overline{\Psi} \Gamma \Psi \right) D \left( \overline{\Psi} \Gamma \Psi \right) \]
Glueballs:
\[ \mathcal{L}_G = \frac{3(3h)^2}{2} \int \delta \phi \ e^{-\mathcal{L}_B[\phi]} \left( \phi \right) D \left( \phi \right) D \left( \phi \right) \]
Baryons:
\[ \mathcal{L}_3 = \frac{g^3 h}{6} \int \delta \phi \ e^{-\mathcal{L}_B[\phi]} \left( \overline{\Psi} \Gamma \Psi \right) D \left( \overline{\Psi} \Gamma \Psi \right) D \left( \overline{\Psi} \Gamma \Psi \right) D \]

3 Two-Quark Bound States

Omitting elaborous details of intermediate calculations, we write shortly only important steps of our approach for describing two-quark (mesonic) stable bound states.
   i. Allocation of one-gluon exchange between quark currents
\[ L_2 = \frac{g^2}{2} \int dx_1 dx_2 \sum_{f_1 f_2} (\overline{q}_{f_1} (x_1) \gamma_\mu t^a q_{f_1} (x_1) D_{\mu \nu}^{ab} (x_1, x_2) (\overline{q}_{f_2} (x_2) \gamma_\nu t^b q_{f_2} (x_2)) \]
   ii. Introduction of the centre-of-masses system:
\[ x_1 = x + \xi_1 y, \quad x_2 = x - \xi_2 y, \quad \xi_1 = \frac{m_{f_2}}{m_{f_1} + m_{f_2}}, \quad \xi_2 = \frac{m_{f_1}}{m_{f_1} + m_{f_2}} \]
   iii. Fierz transformation for colorless bilocal quark currents
\[ i(\gamma_{\mu_1})_{\alpha_1 \beta_1} i(\gamma_{\mu_2})_{\alpha_2 \beta_2} = -1 \cdot S - \frac{1}{2} \cdot V + 0 \cdot T + \frac{1}{2} \cdot A - 1 \cdot P \]
\[ \delta^{\mu \nu} \delta^{\mu \nu'} = \frac{1}{4} \cdot S - \frac{1}{4} \cdot V + \frac{1}{4} \cdot T - \frac{1}{4} \cdot A - \frac{1}{4} \cdot P \]
\[ L_2 = \frac{g^2}{2} \int dy \sum_{f_1 f_2} J_{J_f f_2} (x, y) J_{J_{f_1} f_1} (x, y) \]
   iv. Orthonormal basis \( \{ U_Q(x) \} \) indced by quantum numbers \( Q = \{ nklm \} \):
\[ J_{J_f f_2} (x, y) = \sqrt{D(y)} (\overline{q}_{f_1} (x + \xi_1 y) \Gamma q_{f_2} (x - \xi_2 y)) = \sum_Q J_{J_Q f_1 f_2} (x) U_Q (y) \]
   v. Diagonalization of \( L_2 \) on colorless quark currents:
\[ L_2 = \frac{g^2}{2} \sum_N dx J^+_N(x) J_N(x), \quad N = JQ f_1 f_2, \quad J_{JQ} (x) = \overline{q}_{f_1} (x) V_{JQ} (\overline{\partial}) q_{f_2} (x) \]
vi. Gaussian representation by using auxiliary meson fields:

\[ e^{g^2 (J_N^+J_N)} = \int \int DB_N^+ DB_N^+ e^{-(B_N^+B_N)+g[(B_N^+J_N)+(J_N^+B_N)]} \].

vii. Integration over \( \bar{q}, q \) and introduction of an effective action

\[ S_N[B] = -\frac{1}{2}(B_N^2) + \text{Tr} \ln[1 + g(B_NV_N)S]. \]

viii. Hadronization Ansatz to identify \( B_N(x) \) as meson fields with \( N = \{ JQff \} \).

\[ S_N[B] = -\frac{1}{2}(B_N^2) + \text{Tr} \ln[1 + g(B_NV_N)S], \quad \text{Tr} = \text{Tr}_r \text{Tr}_\gamma. \]

ix. Partition function for mesons and extraction of the full quadratic (kinetic) term:

\[ Z_N = \int DB_N e^{-\frac{1}{2}(B_N^2)+\text{Tr} \ln[1 + g(B_NV_N)S]} = \int DB_N e^{-\frac{1}{2}(B_N[1 + g^2\text{Tr}V_NV_N'S]B_N)+W_I[B_N]} \]

x. The diagonalization of the quadratic form is equivalent to solution of the Bethe-Salpeter equation (in the ladder approximation) on the orthonormal system \( U_N \):

\[ g^2 \text{Tr}(V_NSV_NV_N'S) = (U_N\Pi_pU_{N'}) = \lambda_N(-p^2)\delta_{JJ'}\delta_{QQ'}. \]

xi. The mass spectrum and vertex-functions are defined from:

\[ 1 = \lambda_N(M_N^2), \quad V_N(\bar{\partial}) = \Gamma_J \int dy U_Q(y)\sqrt{D(y)e^{\frac{\Lambda^2}{4}c}}. \]

xii. Below we investigate only two-quark bound states for mesons and will not consider remaining interaction between mesons is described by functional \( W_I[B_N] = O[B_N^3] \).

### 3.1 Bethe-Salpeter Equation for Mesons

The diagonalization of \( g^2 \Pi_p(x, y) \) is equivalent to the solution of the corresponding Bethe-Salpeter equation:

\[ \iint dx dy U_{JQ}(x) g^2 \Pi_p(x, y) U_{J'Q'}(y) = \delta^{QQ'} \delta^{JJ'} \lambda_{JQ}(-p^2) \]

Note, the polarization kernel \( \Pi_p(x, y) \) is real and symmetric that allows one to use a variational technique. Choose a normalized trial wave function (for simplicity consider \( n = 0, l \geq 0 \)) as an approximation for \( U_Q(x) \):

\[ \Psi_{l(\mu)}(x, a) = C_l T_{l(\mu)}(x)\sqrt{D(x)} e^{-\frac{\Lambda^2}{4}c^2}, \]

\[ C_l = \Lambda^{l+1} \sqrt{\frac{(1 + 2c)^{l+1}}{2^l(l + 1)!}}, \quad \sum_{(\mu)} \int dx |\Psi_{l(\mu)}(x, c)|^2 = 1, \]

where \( c > 0 \) - variational parameter; \( T_{l(\mu)}(x) \) - four-dimensional orthogonal polynomials.
3.2 Mass Equation

Variational solution (on the mass shell $p^2 = -M_l^2$) for the Bethe-Salpeter equation for mesons reads:

\begin{equation}
1 = \max_{\Psi_Q} \sum_{\mu} \int dy_1 dy_2 \, \Psi_Q(y_1) \, g^2 \Pi_p(y_1, y_2) \, \Psi_Q(y_2) = \lambda_Q(M_l) \tag{1}
\end{equation}

\begin{align*}
&= \alpha_s \cdot \frac{4}{3\pi} \frac{m_1^2 m_2^2}{m_1^2 + m_2^2} \exp \left\{ \frac{M_l^2 (\mu_1^2 + \mu_2^2) - (m_1^2 + m_2^2)}{2} \right\} \frac{2^l}{(l + 1)!} \\
&\cdot \max_{0 < b < 1} \left\{ \left[ b(1 - b/2) \right]^{l+2} (-1)^l \frac{d^l}{dA^l} \frac{1}{A^2} \exp \left\{ -\frac{M_l^2 (\mu_1 - \mu_2)^2}{4A} \right\} \right. \\
&\left. \cdot \left[ \frac{2\rho}{A} + \frac{M_l^2 (\mu_1 - \mu_2)^2}{4} \left( \frac{2}{A} - \frac{1}{A^2} \right) + M_l^2 \mu_1 \mu_2 + m_1 m_2 (1 + \chi \omega_1 \omega_2) \right] \right\} \\
A &= 1 + 2b, \quad m_i = m_{f_i}/A, \quad M_l = M_{meson}/A
\end{align*}

\[ \rho = \left\{ 1, \frac{1}{2} \right\}, \quad \chi = \{+1, -1\} \quad \text{for} \quad \{P - \text{Pseudoscalar}, V - \text{Vector}\} \]

3.3 Mass-dependent Coupling Constant

Generally, the coupling constant $\alpha_s = g^2/4\pi$ is a free parameter of our model and it can be found as a good fitting parameter for known meson masses. However, it is interesting to consider $\alpha_s$ in dependence on meson masses, for example, by using known results for the QCD running coupling [7, 8]. For this purpose we have tested various versions of such mass dependence (Fig.1):

\[ \frac{1}{\alpha_s(M)} = \frac{1}{\alpha_s(M_0)} + \frac{\beta_0}{4\pi} \ln \left( \frac{M^2}{M_0^2} \right), \quad [\text{QCD asymptotics}] \]

\[ \alpha_s(M) = \frac{4\pi}{\beta_0} \ln \left( \frac{M^2}{Q^2} \right) \left( 1 - \frac{1}{(M^2/Q^2)} \right), \quad [\text{A.Nesterenko(2002)}] \]

\[ \alpha_s(M) = \frac{4\pi}{\beta_0} \left( \frac{1}{\ln(M^2/Q^2)} - \frac{1}{(M^2/Q^2) - 1} \right), \quad [\text{I.Solovtsev, D.V.Shirkov(2003)}] \]

where $\beta_0 = 11 - 2 N_{flav}/3$ and, $\alpha_s(1777\,\text{MeV}) = 0.34$ is known from $\tau$-lepton decay data.
Note, this mass dependence allows one to explain the mass splitting effect between \(\pi(140)\) and \(\eta(547)\) mesons when the quark content is the same for both of them (Fig.2). Besides, the mass-dependent formula due to [7] is found more acceptable for our consideration with \(Q = 600\) MeV.

### 3.4 Ground State: \(l = 0\)

Obtained results of our estimations for \(l = 0\) are shown in Tables 1 and 2, and Figure 2.

![Figure 2: Solutions of the mass equation for \(\pi, \eta\) and \(\eta'\) mesons.](image)

Note, there appears a mass splitting effect between \(\pi(140)\) and \(\eta(547)\) mesons when the quark content is the same for both of them (Fig.2). This is possible only for relatively light quarks \(m_u = m_d = 219\) MeV and the reason is a relevant combination of the decreasing function \(\alpha_s(M)\) and an increasing function standing behind it in (1). But this does not appear for heavier quarks (e.g., \(m_s = 249\) MeV and the unique solution is the meson \(\eta'(958)\)).

<table>
<thead>
<tr>
<th>(J^{PC} = 0^{-+})</th>
<th>(M_P)</th>
<th>(J^{PC} = 0^{++})</th>
<th>(M_P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi(140))</td>
<td>140</td>
<td>(D(1870))</td>
<td>1975</td>
</tr>
<tr>
<td>(\eta(547))</td>
<td>547</td>
<td>(D_s(1970))</td>
<td>1969</td>
</tr>
<tr>
<td>(\eta_c(2979))</td>
<td>2979</td>
<td>(B(5279))</td>
<td>5333</td>
</tr>
<tr>
<td>(\eta_b(9300))</td>
<td>9300</td>
<td>(B_s(5370))</td>
<td>5330</td>
</tr>
<tr>
<td>(K(495))</td>
<td>495</td>
<td>(B_c(6400 \pm 400))</td>
<td>6087</td>
</tr>
</tbody>
</table>

Table 1: Estimated masses for the pseudoscalar mesons.

We see (Table 1) that our model with a set of parameters:
\[
\{\Lambda = 600\text{MeV}, \ m_u = m_d = 219\text{MeV}, \ m_s = 249\text{MeV}, \ m_c = 844\text{MeV}, \ m_b = 4303\text{MeV}\}
\]
leads to quantitatively well description of both light and heavy meson masses.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{$J^{PC} = 1^{-}$} & \textbf{$M_P$} & \textbf{$J^{PC} = 1^{-}$} & \textbf{$M_P$} \\
\hline
\hline
\rho(770) & 770 & K^{*}(892) & 892 \\
\omega(782) & 782 & D^{*}(2010) & 2034 \\
\Phi(1019) & 1019 & D_{s}^{*}(2112) & 2102 \\
J/\Psi(3097) & 3097 & B^{*}(5325) & 5397 \\
\Upsilon(9460) & 9460 & & \\
\hline
\end{tabular}
\caption{Estimated masses for the vector mesons.}
\end{table}

For the vector mesons, the optimal set of parameters is found:
\[ \{ \Lambda = 600 \text{MeV}, \ m_u = m_d = 203 \text{MeV}, \ m_s = 226 \text{MeV}, \ m_c = 887 \text{MeV}, \ m_b = 4387 \text{MeV} \} . \]

3.5 Orbital Excitations: Regge Trajectories

Correct description of the mesonic orbital excitations can serve as an additional testing ground for our model. In fact, the orbital excitations take place in larger distances and therefore, should be less sensitive to the tiny short-range details of the chosen propagators.

The experimental data and our estimates for the $\rho$-meson and $K$-meson families are:
\[
\rho(770) - a_2(1320) - \rho_3(1640) - a_4(2040),
\]
\[
M_1(770) - M_2(1367) - M_3(1702) - M_4(2005),
\]
and
\[
K^{*}(892) - K^{*}_2(1430) - K^{*}_3(1780) - K^{*}_4(2045)
\]
\[
M_1(892) - M_2(1487) - M_3(1791) - M_4(2064) - M_5(2317) - M_6(2551) - M_7(2770)
\]
correspondingly (see Fig.3).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{regge_trajectories.png}
\caption{Regge trajectories of $\rho$- and $K$-meson families}
\end{figure}

As is expected, our model describes well the Regge trajectories of mesons. The most sensitive parameter $\Lambda$ governs the slope of the Regge trajectories and its optimal value $\Lambda = 600 \text{MeV}$ is used in previous section.
4 Glueballs

The experimental status of glueballs is not clear up to now, although they are very interesting objects from the theoretical point of view. Indeed, it is quite intriguing how can massless gluons be merged into massive objects? It is evident that the structure of QCD vacuum plays the main role in formation of glueballs, any direct guess about its explicit structure is avoided.

Omitting details, we just write down the polarization kernel for the $gg$ bound state with quantum numbers $J^{PC} = 0^{++}$:

$$\tilde{\Pi}(p^2) = 6(3h)^2\max\frac{\int d\xi \int d\eta e^{-c(\xi^2 + \eta^2)} D(\xi) D(\eta) \int dze^{-ipz} D\left(z + \frac{\xi - \eta}{2}\right) D\left(z - \frac{\xi - \eta}{2}\right)}{\int d\xi e^{-2c\xi^2} D(\xi)}.$$  

Taking into account:

$$D(\xi) = \frac{\Lambda^2}{(4\pi)^2}e^{-\Lambda^2\xi^2/4}, \quad \kappa = \max\left\{\frac{c + 1/8}{(c + 3/8)(c + 3/8 - 1/48/(c + 3/8))}\right\} = 1.2122...$$

we write the mass equation and obtain the glueball mass:

$$1 - \tilde{\Pi}(-M_G^2) = 0, \quad M_G = \Lambda\sqrt{2\ln\frac{128\pi}{27\lambda\kappa}}.$$  

Substituting the coupling constant and $\Lambda$ we obtain:

$$\lambda = \frac{\hbar^2}{4\pi} = 0.33, \quad \Lambda = 600 \text{ MeV} \quad \Rightarrow \quad M_G = 1614 \text{ MeV}$$

This result may be compared with other predictions of the glueball mass:

- $1730 \pm 130$ MeV M.Peardon [1999]
- $1740 \pm 150$ MeV M.Teper [1999]
- $1580$ MeV Y.A.Simonov [2000]
- $1530 \pm 200$ MeV H.Forkel [2001]

References