CLOTHED PARTICLE REPRESENTATION IN QUANTUM FIELD THEORY: MASS RENORMALIZATION

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Abstract

We consider the neutral pion and nucleon fields interacting via the pseudoscalar Yukawa-type coupling. The method of unitary clothing transformations is used to handle the so-called clothed particle representation, where the total field Hamiltonian and the three boost operators in the instant form of relativistic dynamics take on the same sparse structure in the Hilbert space of hadronic states. In this approach the mass counterterms are cancelled (at least, partly) by commutators of the generators of clothing transformations and the field interaction operator. This allows the pion and nucleon mass shifts to be expressed through the corresponding three-dimensional integrals whose integrands depend on certain covariant combinations of the relevant three-momenta. The property provides the momentum independence of mass renormalization. The present results prove to be equivalent to the results obtained by Feynman techniques.

1 Recollections

Recently [1, 2] the so-called unitary clothing transformation approach has been employed for an approximate treatment of simplest eigenstates of total field Hamiltonian $H$. We mean the physical vacuum $\Omega$ (the lowest-energy $H$ eigenstate) and the observable one-particle states $|p\rangle$ with the momentum $p$. By definition, the vector $|p\rangle$ belongs to the $H$ eigenvalue $E_p = \sqrt{p^2 + m^2}$, where $m$ is the mass of a free particle (e.g., fermion). We call it the physical mass. Thus, $m$ appears here in a natural way via the relativistic dispersion law vs $m$ as a pole of the full particle propagator. In this context, we note the paper [3] where the one-particle energies have been calculated for a nonlocal model of interacting charged and neutral mesons using stationary perturbation theory.

Normally, the mass shifts are expressed through the particle self-energy functions evaluated in nontrivial field theories as expansions in the coupling constants. Miscellaneous self-energy contributions give rise to undesirable divergences. Their removal requires considerable intellectual efforts associated with a consequent regularization of the divergent integrals involved. In the S-matrix calculations they are encountered as early as in the first nonvanishing approximation in the coupling constants. In this connection, note a possible way [4, 5] to express the $S$–matrix directly in terms of renormalized masses and interactions between clothed particles, these quasiparticles within the approach under consideration (cf. [6]).
Regarding the mass renormalization problem we recall one more realization of the unitary transformation method [7, 8], where the Hamiltonian for interacting fields was blockdiagonalized using Okubo’s idea. While in [7] the \( \pi, \rho, \omega \) and \( \sigma \) mesons were coupled with nucleons via the Yukawa-type interactions, the authors of [8] dealt with scalar “nucleons” and mesons with a simpler coupling. This enabled not only to derive the effective (Hermitian and energy independent) interactions (“quasipotentials”) between nucleons, as done in [7], but also to separate the one-nucleon contribution to the Hamiltonian with the renormalized nucleon mass. The authors of [8] have shown that their expression for the second-order nucleon mass shift coincides with the corresponding expression found by Feynman technique. In particular, this shift is independent of the nucleon momentum.

What follows is an extension of the approach [2] to the mass renormalization problem in the clothed particle representation (CPR).

2 Underlying formalism

We proceed with a total Hamiltonian \( H \),

\[
H \equiv H(\alpha) = H_F(\alpha) + H_I(\alpha), \quad H_I(\alpha) = V(\alpha) + M_{\text{ren}}(\alpha),
\]

(1)

where the unperturbed (free) Hamiltonian \( H_F(\alpha) \) and the interaction term \( H_I(\alpha) \) depend on the destruction(creation) operators \( \alpha (\alpha^\dagger) \) of the “bare particles with physical masses” (cf., [2]). \( V(\alpha) \) is the primary interaction between these particles and \( M_{\text{ren}}(\alpha) \) are necessary mass counterterms.

The clothing procedure is aimed at rewriting \( H \) in a new form

\[
H \equiv K(\alpha_c) = K_F(\alpha_c) + K_I(\alpha_c),
\]

(2)

where the free part \( K_F(\alpha_c) \) and the interaction \( K_I(\alpha_c) \) are expressed through the new destruction(creation) operators \( \alpha_c (\alpha_c^\dagger) \) such that

\[
\alpha_c (k, r) \Omega = 0, \quad H \alpha_c^\dagger (k, r) \Omega = k_0 \alpha_c^\dagger (k, r) \Omega, \quad \forall k = (k_0, k), r.
\]

(3)

Here \( \Omega \) denotes the state without physical particles, \( k \) the particle momentum, \( k_0 = \sqrt{k^2 + \mu^2} \), \( \mu \) the physical mass of the particle and \( r \) the polarization index, if any. The “clothed” operators \( \alpha_c \) obey the same algebra as the “bare” operators \( \alpha \) do. One should note that \( K_F(\alpha_c) \neq H_F(\alpha) \) but coincides with \( H_F(\alpha_c) \),

\[
K_F(\alpha_c) = H_F(\alpha_c) = \int dkk_0 \sum_r \alpha_c^\dagger (k, r) \alpha_c (k, r).
\]

(4)

The operator \( K_I(\alpha_c) \) contains the interactions responsible for processes with physical particles. The property of the clothed one-particle states \( \alpha_c^\dagger \Omega \) to be the \( H \) eigenstates is provided if

\[
K_I \alpha_c^\dagger \Omega = 0.
\]

(5)

The clothing itself is implemented via the relation

\[
\alpha (k, r) = W (\alpha_c) \alpha_c (k, r) W^\dagger (\alpha_c), \quad \forall k, r,
\]

(6)
where the unitary transformation (UT)
\[ W(\alpha_c) = W(\alpha) = \exp R(\alpha_c), \quad R^\dagger = -R, \] (7)
removes from \( H \) the so-called 'bad' terms (see [2]).

With the help of (6) we rewrite the total Hamiltonian as
\[ H = H(\alpha) = H(W(\alpha_c) \alpha_c W^\dagger(\alpha_c)) = W(\alpha_c) H(\alpha_c) W^\dagger(\alpha_c) = K(\alpha_c) \]
\[ = H_F(\alpha_c) + H_I(\alpha_c) + [R, H_F] + [R, H_I] + \frac{1}{2}[R, [R, H_F]] + \frac{1}{2}[R, [R, H_I]] + \ldots \] (8)
The operator \( K(\alpha_c) \) is the same Hamiltonian as \( H(\alpha) \) but it has another dependence on its argument \( \alpha_c \) compared to \( H(\alpha) \).

In the Yukawa-type model considered here at the first stage of the clothing procedure to meet the requirement (5) one needs to require (details in [2]):
\[ V = [H_F, R_I], \] (9)
where \( R_I \) is the generator of the first clothing UT \( W_1 = \exp R_I \). Doing so, we find
\[ K(\alpha_c) = H_F(\alpha_c) + M_{\text{ren}}(\alpha_c) + \frac{1}{2}[R_I, V] + \frac{1}{2}[R_I, M_{\text{ren}}] + \frac{1}{3}[R_I, [R_I, V]] + \ldots . \] (10)
The four-operator \( (g^2\text{-order}) \) interactions between clothed particles stem from \( \frac{1}{2}[R_I, V] \) (see, e.g., [9, 10]). The two-operator contributions to it can be compensated by \( M_{\text{ren}}(\alpha_c) \), bringing the definition of the particle mass shifts in the \( g^2\text{-order} \). The r.h.s. of Eq. (10) embodies other bad terms of the \( g^2\text{- and higher orders, which can be eliminated in the same way via the subsequent UT's.} \)

3 Clothing procedure in action. Cancellation of mass counterparts

In the following model, where a neutral spinor (fermion) field \( \psi \) interacts with a neutral pseudoscalar meson field \( \phi \) by means of the Yukawa coupling, \( H \) can be expressed through bare destruction (creation) operators \( a(k) \left( a^\dagger(k) \right), b(p, r) \left( b^\dagger(p, r) \right) \) and \( d(p, r) \left( d^\dagger(p, r) \right) \) for the meson, the fermion and the antifermion, respectively. Here \( k \) and \( p \) are the particle momenta. In fact, we have its free part
\[ H_F \equiv H_F(\alpha) = \int d\mathbf{k} \omega_k a^\dagger(k) a(k) + \int d\mathbf{p} E_\mathbf{p} \left[ b^\dagger(p, r) b(p, r) + d^\dagger(p, r) d(p, r) \right], \] (11)
and the primary interaction
\[ V(\alpha) = \int d\mathbf{k} \hat{V}^k a(k) + H.c., \quad \hat{V}^k = \int d\mathbf{p}' d\mathbf{p} \sum_{r, r'} F^\dagger_{r, r'}(\mathbf{p}', \mathbf{r}') \ V^k_{r, r'}(\mathbf{p}', \mathbf{r}; \mathbf{p}, \mathbf{r}) F(\mathbf{p}, \mathbf{r}), \] (12)
where operator column \( F \) and row \( F^\dagger \) are composed of the bare nucleon and antinucleon operators (e.g., \( F^\dagger(\mathbf{p}, \mathbf{r}) \equiv \left[ b^\dagger(\mathbf{p}, \mathbf{r}), d(-\mathbf{p}, \mathbf{r}) \right] \)), and we have introduced the c-number matrices
\[ V^k_{r, r'}(\mathbf{p}', \mathbf{r}; \mathbf{p}, \mathbf{r}) = \begin{bmatrix} V^k_{11}(\mathbf{p}', \mathbf{r}; \mathbf{p}, \mathbf{r}) & V^k_{12}(\mathbf{p}', \mathbf{r}; \mathbf{p}, \mathbf{r}) \\ V^k_{21}(\mathbf{p}', \mathbf{r}; \mathbf{p}, \mathbf{r}) & V^k_{22}(\mathbf{p}', \mathbf{r}; \mathbf{p}, \mathbf{r}) \end{bmatrix} \]
\[
\delta\mu^2 = \frac{2g^2}{(2\pi)^3} \int \frac{dp}{E_p} \left\{ \frac{p_- k}{\mu^2 + 2p_- k} - \frac{pk}{\mu^2 - 2pk} \right\} = \frac{2g^2}{(2\pi)^3} \int \frac{dp}{E_p} \left\{ 1 + \frac{\mu^4}{4(pk)^2 - \mu^4} \right\},
\]

i.e., it is independent of the meson momentum \(k\). Here we have introduced the 4-vectors \(p = (E_p, p), p_- = (E_p, -p)\) and \(k = (\omega_k, k)\).
In the course of our consideration, the second-order contributions to the fermion mass counterterm are cancelled by the following two-operator combination,
\[
\frac{1}{2} [R, V]_{2\text{ferm}} = \int d\mathbf{k} F^i_c X^k F_c = \int d\mathbf{k} \left\{ b^i_c X^k_{11} b_c + b^i_c X^k_{12} d^i_c + d_c X^k_{21} b_c + d_c X^k_{22} d^i_c \right\}. \tag{19}
\]
Explicit expressions for the c-number matrix elements \(X^k\) in terms of \(V^k\) and \(R^k\) can be found in [11].

First of all, we are interested in cancellation of the \(b^i b_c\) and \(d_c d^i\) to get a prescription in determining the fermion (nucleon) mass renormalization (of course, in the \(g^2\)-order). To this end, we assume
\[
m\delta m^{(2)} M_{11} + \int d\mathbf{k} X^k_{11} = 0, \quad m\delta m^{(2)} M_{22} + \int d\mathbf{k} X^k_{22} = 0, \tag{20}\]
or in spinor space,
\[
m\delta m^{(2)} \frac{\delta (\mathbf{p}' - \mathbf{p})}{E_p} \delta_{rr'} = -\int d\mathbf{k} X^k_{11} (\mathbf{p}', r'; \mathbf{p}, r),
\]
\[
m\delta m^{(2)} \frac{\delta (\mathbf{p}' - \mathbf{p})}{E_p} \delta_{rr'} = \int d\mathbf{k} X^k_{22} (\mathbf{p}', r'; \mathbf{p}, r). \tag{21}\]

After this all we need is to prove that each of these integrals depend on fermion momentum and spin as \(C(p) \delta (\mathbf{p}' - \mathbf{p}) \delta_{rr'}/E_p\) and show that \(C(p)\) is a constant. As shown in [11],
\[
\int d\mathbf{k} X^k (\mathbf{p}', r'; \mathbf{p}, r) = -\frac{g^2}{4(2\pi)^3} \frac{\delta (\mathbf{p}' - \mathbf{p})}{E_p} \delta_{rr'} I(p), \tag{22}\]
where
\[
I(p) = \int \frac{d\mathbf{k}}{E_p - E_{\mathbf{k}} - \omega_k} \left\{ \frac{m^2 - \mathbf{p} \cdot \mathbf{p}}{E_p - E_{\mathbf{k}} - \omega_k} - \frac{m^2 + \mathbf{p} \cdot \mathbf{p}}{E_p + E_{\mathbf{k}} + \omega_k} \right\}. \tag{23}\]

After some transformations we find,
\[
I(p) = I_1(p) + I_2(p),
\]
\[
I_1(p) = \int \frac{d\mathbf{k}}{\omega_k} \frac{1}{\omega_k} \left\{ \frac{1}{\mu^2 - 2pk} - \frac{1}{\mu^2 + 2pk} \right\},
\]
\[
I_2(p) = \int \frac{d\mathbf{q}}{E_q} \left\{ \frac{m^2 - pq}{2[m^2 - pq]} + \frac{m^2 + pq}{2[m^2 + pq]} - \mu^2 \right\}.
\]

Thus, mass shift of interest is
\[
\delta m^{(2)} = \frac{g^2}{4m(2\pi)^3} I(p) = \frac{g^2}{4m(2\pi)^3} [I_1(m, 0, 0, 0) + I_2(m, 0, 0, 0)]. \tag{24}\]
The second relation (21) leads to the same result since \(X^k_{22} = -X^k_{11}\). The integrals involved in Eq. (24) can be reduced to the elementary ones. Remaining crossed \(b^i d^i\) and \(d_c b_c\) terms in Eq. (19) are bad having nonvanishing matrix elements between the vacuum \(\Omega\) and two-fermion states. It turns out that they are not covariant and should be removed by means of a consequent UT linear in them. Thus, unlike the meson mass renormalization only the particle-conserving part of the nucleon mass counterterm (responsible for one fermion \(\rightarrow\) one fermion transition) may be cancelled via one and the same clothing UT.
4 Comparison with an explicitly covariant calculation. Elimination of divergences in the S-matrix

The considered procedure enables us to remove from the Hamiltonian in CPR not only “bad” terms. Simultaneously, “good” two-particle terms are eliminated too being compensated with corresponding mass counterterms. Along the guideline some ultraviolet divergences inherent in the conventional form of $H$ cannot appear in the $S$-matrix. In the context, let us recall Dyson expansion for the $S$ operator,

$$S = 1 - i \int_{-\infty}^{\infty} dt H_I(t_1) + (i)^2 \frac{1}{2!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 P[H_I(t_1) H_I(t_2)] + ...,$$  \hspace{1cm} (25)

where, as usually, $H_I(t) = \exp[iH_F t] H_I(a) \exp[-iH_F t]$ is an interaction in Dirac ($D$) picture. To be definite, we consider the interacting neutral pion and nucleon fields with the operator $H_I(\alpha) = V(\alpha) + M_{ren}(\alpha)$ (see Eqs.(12) and (17)) and matrix elements $\langle f | S^{(2)} | i \rangle$ of the $S$ operator in $g^2$-order, sandwiched between initial and final $\pi^0 N$ states,

$$|i\rangle = a^\dagger (k) b^\dagger (p, r) \Omega_0, \hspace{1cm} |f\rangle = a^\dagger (k') b^\dagger (p', r') \Omega_0.$$  \hspace{1cm} (26)

We are interested in competition between fermion mass renormalization contribution to $S^{(2)}$ and the so-called fermion self-energy diagram contribution:

$$\langle f | S^{(2)}_{SE} | i \rangle = -\frac{g^2}{(2\pi)^3 m^2} I_F(p) \left[ \frac{\delta(p' - p)}{E_p} \delta(k' - k) \delta(E_{p'} + \omega_{k'} - E_p - \omega_k) \delta_{rr} \right],$$  \hspace{1cm} (27)

$$I_F(p) = \int dq \frac{d^4 q}{q^2 - \mu^2 + i0} \left\{ 1 - \frac{p(p - q)}{m^2} \right\} \frac{1}{(p - q)^2 - m^2 + i0},$$

or

$$I_F(p) = \int dq \int_{-\infty}^{\infty} dq_0 \frac{1}{q^2 - \omega_q^2 + i0} \left\{ 1 - \frac{p_0(p_0 - q_0) - p(q - q)}{m^2} \right\} \frac{1}{(p_0 - q_0)^2 - E_{p - q}^2 + i0}.$$  \hspace{1cm}

The ”forward-scattering” process associated with this diagram would be responsible for appearance of certain infinity in $\pi^0 N$ scattering amplitude $\langle f | T | i \rangle$. Following a common practice, the divergence should be compensated by $\langle f | M^{(2)}_{ferm}(\alpha) | i \rangle$ piece, viz., it is required that

$$2\pi i \langle f | M^{(2)}_{ferm}(\alpha) | i \rangle \delta(E_f - E_i) = \langle f | S^{(2)}_{SE} | i \rangle .$$  \hspace{1cm} (28)

At this point, one should emphasize that similar well-known steps become unnecessary if from the beginning we operate with clothed particle representation $K(\alpha_c)$ of Hamiltonian $H(\alpha)$. This new form of $H$ does not contain ultraviolet divergences and, being constructed via sequential unitary transformations, gives new unitarily equivalent forms of the $S$ operator (see [4, 5]). It is important that the approach enables us to evaluate one and the same $S$ matrix with nonperturbative methods.

Now, by taking into account pole disposition for propagators involved and carrying out $q_0$-integration, one can get,

$$\langle f | S^{(2)}_{SE} | i \rangle = \frac{\pi i g^2}{2 (2\pi)^3} \frac{\delta(p' - p)}{E_p} \delta(k' - k) \delta(E_{p'} + \omega_{k'} - E_p - \omega_k) \delta_{rr}.$$
\[
\int \frac{dq}{E_{p-q} \omega_q} \left\{ \frac{m^2 - E_p E_{p-q} + p(p - q)}{E_p - E_{p-q} - \omega_q} - \frac{m^2 + E_p E_{p-q} + p(p - q)}{E_p + E_{p-q} + \omega_q} \right\}.
\] (29)

The three-dimensional integral in (29) coincides with integral \( I(p) \) defined by Eq. (23). Hence, one can write

\[
\langle f | S_E^{(2)} | i \rangle = \pi i \frac{g^2}{2(2\pi)^3} I(p) \frac{\delta(p' - p)}{E_p} \delta(k' - k) \delta(E_{p'} + \omega_{k'} - E_p - \omega_k) \delta_{r'r}.
\] (30)

It follows from (27) and (30) that

\[
I_F(p) = -\frac{\pi i}{2m^2} I(p),
\] (31)
i.e., we have another proof of the \( p \)-independence of \( I(p) \) since \( I_F(p) \) is an explicitly covariant quantity. Besides, we have expressed the Feynman one-loop integral \( I_F(p) \) through other covariant integrals \( I_1(p) \) and \( I_2(p) \).

5 Some general links

Let us consider the momentum independence in question from a general point of view, viz., for the one-particle matrix elements

\[
\langle k' | S | k \rangle = \langle k' | [1 + S^{(1)} + S^{(2)} + ...] | k \rangle
\]
to be definite between the spinless (pion) states \( |k\rangle = a^\dagger(k) |\Omega_0\rangle \). We are interested in

\[
\langle k' | S^{(2)} | k \rangle = -2\pi i \delta(\omega_{k'} - \omega_k) \langle k' | T^{(2)}(\omega_k) | k \rangle
\]
with the second order \( T \)-operator

\[
T^{(2)}(\omega_k) = V(\omega_k + i0 - H_F)^{-1} V
\]

To set links with previous results it is sufficient to note that

\[
\frac{1}{2} \langle k' | [R_1, V] | k \rangle = \langle k' | V(\omega_k + i0 - H_F)^{-1} V | k \rangle
\] (32)

if pion mass \( \mu < 2m \). In particular, it means that within the considered model for \( V \) the propagator with intermediate nucleon-antinucleon states in Eq. (32) is not singular. Then, according to [2], the generator \( R_1 \) is

\[
R_1 = -i \lim_{\varepsilon \to 0^+} \int_0^\infty dt V_D(t) e^{-\varepsilon t},
\]

and proof of Eq. (32) is trivial. Here, as usually, \( V_D(t) = \exp[iH_F t] V \exp[-iH_F t] \).

Using translational invariance of \( V \), one can show that

\[
\langle k' | V(\omega_k + i0 - H_F)^{-1} V | k \rangle = \frac{\delta(k' - k)}{\omega_k} G(k),
\]

7
where $G(k)$ is a function of the four-momentum $k = (\omega_k, \mathbf{k})$. Indeed, putting $V = \int d\mathbf{x} V(\mathbf{x})$ with the interaction density $V(\mathbf{x})$ in the $D$ picture being the Lorentz scalar

$$U(\Lambda) V_D(x) U^{-1}(\Lambda) = V_D(\Lambda x),$$

and using

$$(\omega_k + i0 - H_F)^{-1} = -i \lim_{\epsilon \to 0^+} \int_0^\infty dt e^{i(\omega_k + i\epsilon - H_F)t},$$

we arrive at

$$G(k) = -i (2\pi)^3 \frac{1}{2} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dt e^{-\epsilon t} \int d\rho \langle 0 | a(k) V_D \left( \frac{1}{2} \rho \right) V_D \left( -\frac{1}{2} \rho \right) a^\dagger(k) | 0 \rangle.$$  

Here, as in [2], we are addressing operators $a(k) = \sqrt{\omega_k} a(k)$ that meet the covariant commutation rules

$$[a(k), a^\dagger(k')] = \omega_k \delta(k' - k).$$

It results in appearance of a typical combination

$$-2\pi i \frac{\delta(\omega_{k'} - \omega_k) \delta(k' - k)}{\omega_k} G(k)$$

in the correspondent $S$-matrix element. Thus, $G(k)$ is independent on $k$. At this point, let us recall the relativistic invariance property

$$\frac{\langle k' | S | k \rangle}{\sqrt{\omega_k \omega_{k'}}} = \frac{\langle \Lambda k' | S | \Lambda k \rangle}{\sqrt{\omega_{k'} \omega_{k'}}}.$$  

In its turn, the meson mass shift can be connected with the $c$-number $G(k) = G(\mu, 0, 0, 0)$ in evaluating the one-meson matrix elements in the l.h.s. of Eq. (32).

This consideration gives us a possible (probably, general) way when finding the momentum independence of mass shifts within this three-dimensional formalism, at least, in the first nonvanishing order in coupling constant.

6 Summary

We have demonstrated here how the mass shifts in the system of interacting pion and nucleon fields can be calculated by the use of the clothed particle representation. The respective mass counterterms are compensated and determined directly in the Hamiltonian.

The procedure described above has an important feature, viz., the mass renormalization is made simultaneously with the construction of a new family of quasipotentials (Hermitian and energy independent) between the physical particles (the quasiparticles of the method). Explicit expressions for the quasipotentials can be found in [2, 9].

By using a comparatively simple analytical means, we could show that the three-dimensional integrals, which determine the pion and nucleon renormalizations in the second order in the coupling constant $g$, can be written in terms of the Lorentz invariants.
composed of the particle three-momenta. In other words, these integrals are independent of the particle momentum.

The experience acquired has allowed us, on the one hand, to reproduce the manifestly covariant result by Feynman techniques and, on the other hand, to derive a new representation for the Feynman integral that corresponds to the fermion self-energy diagram. Of course, here we are dealing with the coincidence of the two divergent quantities: one of them is determined by the nucleon mass renormalization one-loop integral, while the other stems from the commutator \([R, V]\). We are trying to overcome this drawback by means of the introduction of the cutoff functions in momentum space. Such functions have certain properties to do the theory to be satisfied the basic symmetry requirements.

References